Rings with additive group which is a torsion-free group of rank two

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I. Introduction

In the past ten years considerable progress has been made in the construction of rings with given additive group and the related problem of characterizing the additive groups of rings satisfying various conditions. A complete bibliography of these results is given at the end of this paper.

Given an abelian group G, then we call $\mathfrak{R}$ *a ring over* G if the additive group $\mathfrak{R}^+ = G$. Throughout this paper *ring* means *associative ring*.

The results for rings over torsion-free abelian groups are meager. The rings over a given torsion-free group of rank 1 have been determined, and every such ring is either a zero-ring ($XY=0$ for $X, Y \in \mathfrak{R}$) or is isomorphic to a subring of the field R of rational numbers (RÉDEI and SZELE [6], and BEAUMONT and ZUCKERMAN [2]). SZELE [9] has given a sufficient condition that an arbitrary torsion-free group be a nil group, and REE and WISNER [7] have found necessary and sufficient conditions that a completely reducible torsion-free group be a nil group. BEAUMONT [1] gave a construction which included all rings over free abelian groups, and FUCHS [3] extended this construction to divisible torsion-free groups. RÉDEI [5] generalized [1] to obtain algebras with given additive module.

In the present work, we consider rings over torsion-free abelian groups of rank 2 and the principal results are summarized below. Except where otherwise explicitly stated, the group G will always be a torsion-free abelian group of rank 2. In Section III, a characterization of a group G in terms of groups of rank 1 is given. This characterization depends on the choice of the rational basis for G. In Section IV, we find a necessary and sufficient condition that there exist a non-commutative ring over G and determine all

*) Supported in part by the Haverford College Faculty Research Fund.

such rings (Theorem 2). It follows from the latter result that every ring without zero-divisors over G is commutative, and in Section V we prove (Theorem 3) that $\mathfrak{R}$ is a ring without zero-divisors over G if and only if $\mathfrak{R}$ is isomorphic to a subring of a quadratic field extension $R(\alpha)$ of the rationals R.

II. Rational operators

Let H be a torsion-free abelian group, let $X_1, X_2, \ldots, X_n$ be elements of H, and let $a_1/b_1, a_2/b_2, \ldots, a_n/b_n$ be rational numbers. Denote the least common multiple of the integers b_i $(i=1,2,\ldots,n)$ by $[b_i]$. If the equation

$$(1) \qquad [b_i]X = \sum_{j=1}^{n} [b_i](a_j/b_j)X_j$$

has a solution $X \in H$, then this solution is unique and we write

$$(2) \qquad X = \sum_{j=1}^{n} (a_j/b_j)X_j.$$

Let $X = \sum_{j=1}^{n} r_j X_j$ and $Y = \sum_{j=1}^{n} s_j X_j$, where r_i and s_i $(i=1,2,\ldots,n)$ are rational numbers, be elements of H as described above. Then it is routine to check that

$$(3) \qquad X \pm Y = \sum_{j=1}^{n} (r_j \pm s_j)X_j.$$

Further, if $\mathfrak{R}$ is a ring with H as its additive group, the distributive laws in $\mathfrak{R}$ yield

$$(4) \qquad X \cdot Y = \sum_{i,j=1}^{n} r_i s_j (X_i \cdot X_j).$$

If the elements $X_1, X_2, \ldots, X_n$ are independent elements of H, and if $X = \sum_{j=1}^{n} r_j X_j$ with r_j $(j=1,2,\ldots,n)$ rational numbers, then this representation is unique. Denote the pure subgroup of H generated by $X_1, X_2, \ldots, X_n$ by $\{X_1, X_2, \ldots, X_n\}$. Then the mapping $X \to (r_1, r_2, \ldots, r_n)$ is an isomorphism of $\{X_1, X_2, \ldots, X_n\}$ onto a subgroup of the divisible torsion-free group of rank n. In other words, $\{X_1, X_2, \ldots, X_n\}$ is isomorphic to a subdirect sum of groups of rank 1. If H has rank n, then $H = \{X_1, X_2, \ldots, X_n\}$.

III. Characterization of groups of rank 2

For the purposes of this paper, it will be convenient to give a characterization of torsion-free abelian groups of rank 2 in terms of certain groups of rank 1. Let X_1, X_2 be independent elements of a group G of rank 2, so that $G=\{X_1, X_2\}$ in the terminology of the preceding paragraph. Each element $X \in G$ has the unique representation $X=uX_1+vX_2$, where u, v are rational numbers, and let U and V be the subgroups of R^+ determined by the projections $X \to u$ and $X \to v$, respectively, for every $X \in G$.

Let $U_0=[u \in U | uX_1 \in G]$ and $V_0=[v \in V | vX_2 \in G]$. Then U_0 and V_0 are subgroups of U and V, respectively, which are isomorphic to the pure subgroups $\{X_1\}$ and $\{X_2\}$ of G.

We call $U \supseteq U_0, V \supseteq V_0$, *the groups of rank* 1 *belonging to the independent set* X_1, X_2 *of* G.

Theorem 1. *Let $G=\{X_1, X_2\}$ be a torsion-free abelian group of rank 2. If U, U_0, V, V_0 are the groups of rank 1 belonging to X_1, X_2, then $U/U_0 \cong V/V_0$. Conversely, given the groups of rank 1, $U \supseteq U_0$, $V \supseteq V_0$ such that $U/U_0 \cong V/V_0$, then there exists a group G of rank 2 with independent elements X_1, X_2 such that U, U_0, V, V_0 are the groups belonging to X_1, X_2. G is essentially uniquely determined by U, U_0, V, V_0, and the isomorphism between U/U_0 and V/V_0.*

Proof. (a) Let $G=\{X_1, X_2\}$ be given. Then for $u \in U$, there exists an element $X=uX_1+vX_2$ in G. The isomorphism between U/U_0 and V/V_0 is given by the correspondence $u+U_0 \to v+V_0$.

(b) Let groups of rank 1, $U \supseteq U_0$, $V \supseteq V_0$ be given such that θ: $U/U_0 \to V/V_0$ is an isomorphism onto. Define $G=[(u, v) | u \in U, v \in \theta(u+U_0)]$ where equality and addition are defined componentwise. If $(u, v) \in G$ and $(u', v') \in G$, then $u+u' \in U, v \in \theta(u+U_0)=v+V_0$ and $v' \in \theta(u'+U_0)=v'+V_0$. Hence $v+v' \in v+v'+U_0=\theta(u+U_0)+\theta(u'+U_0)=\theta(u+u'+U_0)$, so that $(u+u', v+v') \in G$. Since the components u and v of the pairs $(u, v) \in G$ are elements of the abelian groups U and V respectively, the operation is associative and commutative. The element $(0, 0) \in G$ is the identity. If $(u, v) \in G$, then $u \in U$ and $v \in \theta(u+U_0)$, so that $-u \in U$ and $-v \in -\theta(u+U_0)=$ $=\theta(-u+U_0)$, and it follows that $(-u, -v) \in G$. Thus G is an abelian group.

Now suppose $n(u, v)=(nu, nv)=(0, 0)$ where n is a positive integer. Since $u \in U, v \in V$, and U and V are torsion-free, this implies $u=0$ and $v=0$. Hence G is torsion-free.

Since $U_0 \neq 0$, $V_0 \neq 0$ (each having rank 1), there exists $u_0 \neq 0$ in U_0 and $v_0 \neq 0$ in V_0. Then $(u_0, 0) \in G$ since $0 \in \theta(u_0+U_0)=\theta(U_0)=V_0$. Similarly, $(0, v_0) \in G$. If $m(u_0, 0)+n(o, v_0)=(mu_0, nv_0)=(0, 0)$ for integers m

and n, then $m=n=0$ since U_0 and V_0 are torsion-free. Hence the elements $(u_0, 0)$, $(0, v_0)$ are indepedent. For $(u, v) \in G$, there exist integers a, b, c, d such that $au=bu_0$ and $cv=dv_0$ since U and V have rank 1. Then $ac(u, v)=(acu, acv)=(bcu_0, adv_0)=bc(u_0, 0)+ad(0, v_0)$. Thus G has rank 2.

(c) Suppose now that φ: $U \to U'$ and χ: $V \to V'$ are isomorphisms onto U' and V', respectively. Let $\varphi(U_0)=U'_0$, $\chi(V_0)=V'_0$, and let $\bar{\varphi}$ and $\bar{\chi}$ be the induced maps $\bar{\varphi}$: $U/U_0 \to U'/U'_0$, $\bar{\chi}$: $V/V_0 \to V'/V'_0$. Further let ψ and ψ' be isomorphisms such that the following diagram is commutative:

$$\begin{array}{ccc} U/U_0 & \xrightarrow{\psi} & V/V_0 \\ {\scriptstyle\bar{\varphi}}\downarrow & & \downarrow{\scriptstyle\bar{\chi}} \\ U'/U'_0 & \xrightarrow{\psi'} & V'/V'_0. \end{array}$$

Consider the groups $G=\{X_1, X_2\}$ and $G'=\{X'_1, X'_2\}$ such that U, U_0, V, V_0 and U', U'_0, V', V'_0 belong to X_1, X_2 and X'_1, X'_2, respectively. The correspondence ω given by $\omega(u, v)=(\varphi(u), \chi(v))$ is a mapping $G \to G'$. For $\varphi(u) \in U'$ and since $v \in \psi(u+U_0)$, $\chi(v) \in \bar{\chi}\psi(u+U_0)=\psi'\bar{\varphi}(u+U_0)=\psi'(\varphi(u)+U'_0)$, so that $(\varphi(u), \chi(v)) \in G'$. It is easy to verify that ω is an isomorphism of G onto G' such that $\omega(X_1)=X'_1$ and $\omega(X_2)=X'_2$. Hence G as constructed in (b), is essentially uniquely determined.

IV. Non-commutative rings over G

In this section we find a necessary and sufficient condition that there exist a non-commutative ring over G and determine all such rings.

Let X be an element of a torsion-free abelian group H, and let R_X be the subgroup of R^+ which is naturally isomorphic to the subgroup $\{X\} \subseteq H$, that is, R_X is the set of rational numbers r such that $rX \in H$.

Definition. The *nucleus* D of H is the subgroup of R^+ defined by $D=\bigcap_{X \in H} R_X$. The proof of the main theorem follows easily from several preliminary lemmas.

Lemma 1. *Let $\mathfrak{R}$ be a ring over a torsion-free group G of rank 2. If $\mathfrak{R}$ is non-commutative, then the elements Z and Z^2 are dependent elements of G for every $Z \in \mathfrak{R}$. If $\mathfrak{R}$ is commutative and if $\mathfrak{R}$ contains an element X such that $X^2 \neq 0$, then there exists an element $Z \in R$ such that Z and Z^2 are independent elements of G.*

Proof. Suppose first that $\mathfrak{R}$ is non-commutative and let $Z \in \mathfrak{R}$. If Z and Z^2 were independent, then every element of G could be written $rZ+sZ^2$ for rationals r, s, and this implies that $\mathfrak{R}$ is commutative by (4), Section II.

Now suppose that $\mathfrak{R}$ is commutative and that $\mathfrak{R}$ contains an element X such that $X^2 \neq 0$. Assume that Z and Z^2 are dependent for every $Z \in \mathfrak{R}$. Let X and Y be independent elements of G. Then we have $X^2 = rX$, $Y^2 = sY$, $XY = YX = tX + uY$ for rationals r, s, t, u, where we may assume that $r \neq 0$. We obtain $X^2Y = rXY$ and $X^2Y = tX^2 + uXY = rtX + uXY$. Hence $(r-u)XY = rtX$, and we consider two cases:

Case I: $r = u$. Here $t = 0$ and $XY = YX = uY = rY$. We have $(X+Y)^2 = rX + rY + rY + sY$. By hypothesis, $(X+Y)^2 = a(X+Y)$ for some rational number a. Hence $(r-a)X + (2r+s-a)Y = 0$, and this implies $r = a, r+s = 0$ Similarly $(X-Y)^2 = rX - rY - rY + sY$ and $(X-Y)^2 = b(X-Y)$ yields $r = b, r = s$. Hence $r+s = 2r = 0$, which is a contradiction.

Case II: $r \neq u$. Here $(r-u)XY = rtX$ combined with $(r-u)XY = (r-u)tX + (r-u)uY$ yields $u = 0$. From $Y^2X = sYX$ and $Y^2X = tYX + uY^2 = tYX$, we obtain either $s = t$ in which case $XY = YX = sX$, or $XY = YX = 0$. As in Case I, the computation of $(X+Y)^2$ and $(X-Y)^2$ yields $r = 0$, whichever of the alternatives holds.

Hence in each case, the assumption that Z and Z^2 are dependent for every $Z \in \mathfrak{R}$ leads to a contradiction, and this completes the proof of the lemma.

Lemma 2. *Let $\mathfrak{R}$ be a non-commutative ring over a torsion-free group G of rank 2. Then there exist independent elements $X, Y \in G$ and a rational number $a \neq 0$ in the nucleus D of G such that X and Y satisfy one of the following multiplication tables:*

$$X^2 = aX, XY = aY, YX = 0, Y^2 = 0; \tag{5}$$

$$X^2 = aX, XY = 0, YX = aY, Y^2 = 0. \tag{5'}$$

Proof. By Lemma 1, for every $Z \in G$, the elements Z and Z^2 are dependent elements of G. Let $G = \{X, Y\}$ where X and Y are independent elements of G, let $X^2 = aX$, $Y^2 = bY$, $XY = cX + dY$, $YX = eX + fY$, where a, b, c, d, e, f are rational multipliers. Let $K, L \in \mathfrak{R}$, where $K = mX + nY$ and $L = sX + tY$. By (4), Section II,

$$KL = msX^2 + mtXY + nsYX + ntY^2.$$

If $a = b = 0$, then $KL = mtXY + nsYX$, and $\mathfrak{R}$ would have a trivial commutative multiplication if $XY = YX = 0$. Hence not both $XY = 0$ and $YX = 0$, say $XY \neq 0$. Then $0 = XY^2 = cXY + dY^2 = cXY$ implies $c = 0$, and $0 = X^2Y = dXY$ implies $d = 0$. But then $XY = cX + dY = 0$, which is a contradiction. Similarly $YX \neq 0$ leads to a contradiction.

Hence we may assume not both $a=0$ and $b=0$ and we consider the case $a \neq 0, b=0$. By calculating XY^2, Y^2X, X^2X, and YX^2 we find $c=e=f=0, a=d$. Thus we have

$$X^2=aX, \quad Y^2=0, \quad XY=aY, \quad YX=0.$$

If $a=0, b \neq 0$, we similarly obtain

$$X^2=0, \quad Y^2=bY, \quad XY=0, \quad YX=bX.$$

Suppose now that $a \neq 0, b \neq 0$. If $XY=0$ then $0=(XY)X=X(YX)=$ $=eX^2+fXY=eX^2$ implies $e=0$, and $0=Y(XY)=(YX)Y=eXY+$ $+fY^2=fY^2$ implies $f=0$, so that $YX=eX+fY=0$. But with $XY=YX=0$, $\mathfrak{R}$ would be commutative. Hence either $c \neq 0$ or $d \neq 0$. Since $aXY=$ $=X^2Y=X(XY)=cX^2+dXY=acX+dXY$, we have $(a-d)XY=acX$. Now $a=d$ implies $c=0$, and $a \neq d$ implies $d=0$ since otherwise there would be a dependency between X and Y. Thus we have two cases to consider.

Case I: $c=0, d \neq 0$. Then $a=d$ and $XY=aY$

Case II: $c \neq 0, d=0$. Then $bXY=XY^2=(XY)Y=cXY$ implies $b=c$. so that $XY=bX$.

By an analysis similar to the above, we can show that either $YX=bX$ or $YX=aY$. Thus there are two apparent cases where $\mathfrak{R}$ is not commutative.

(i) $$X^2=aX, \quad XY=aY, \quad YX=bX, \quad Y^2=bY$$

(ii) $$X^2=aX, \quad XY=bX, \quad YX=aY, \quad Y^2=bY.$$

Now let $Z=XY-YX$. If (i) holds, then $Z=aY-bX$ and X and Z are independent elements of G such that

(5) $$X^2=aX, \quad XZ=aZ, \quad ZX=0, Z^2=0, \quad a \neq 0.$$

If (ii) holds, $Z=bX-aY$ and we have

(5′) $$X^2=aX, \quad XZ=0, \quad ZX=aZ, \quad Z^2=0.$$

We complete the proof of the lemma by showing that $a \in D$. Let $mX+nZ$ be an arbitrary element of G. Then by (5) $X(mX+nZ)=mX^2+$ $+nXZ=maX+naZ=a(mX+nZ) \in G$. Hence $a \in D$. Similarly, if (5′) holds, $(mX+nZ)X=maX+naZ=a(mX+nZ) \in G$.

Lemma 3: *Let $\mathfrak{R}$ be a ring over a torsion-free group G of rank 2 such that there exist independent elements $X, Y \in G$ which satisfy (5) or (5′) of Lemma 2, and let U be the group of rank 1 belonging to X. Then $aU \subseteq D$.*

Proof. Let $r \in U$. Then there exists $Z \in G$ such that $Z = rX + sY$. Let $W = eX + fY$ be an arbitrary element of G. Then by (5), $ZW = rfaX + rfaY = arW \in G$. Hence $ar \in D$ for all $r \in U$. Similarly by (5'), $WZ = arW \in G$,

Corollary. *Under the hypotheses of Lemma* 3, *U is isomorphic to D.*

Proof. Since for $a \neq 0$, aU is isomorphic to U, it follows from Lemma 3 that U is isomorphic to a subgroup of D. On the other hand, $D \subseteq R_X = U_0 \subseteq U$, so that D is isomorphic to a subgroup of U. Then by [4; p. 210], U is isomorphic to D.

Theorem 2. *Let G be a torsion-free group of rank* 2. *Then $\mathfrak{R}$ is a non-commutative ring over G if and only if multiplication in G is defined by $XY = \xi(X)Y$ or $XY = \xi(Y)X$, for $X, Y \in G$, where ξ is a non-trivial homomorphism of G into the nucleus D of G.*

Proof. If $\mathfrak{R}$ is a non-commutative ring over G, then by Lemma 2, $G = \{X_1, X_2\}$ where X_1, X_2 satisfy (5) or (5'). Suppose (5) is satisfied. For $rX_1 + sX_2,\ mX_1 + nX_2 \in G$, we have

$$(rX_1 + sX_2)(mX_1 + nX_2) = rmaX_1 + rnaX_2 = ra(mX_1 + nX_2).$$

It follows from Lemma 3 that the mapping $\xi\colon G \to D$ defined by $\xi(rX_1 + sX_2) = ra$ is a non-trivial homomorphism of G into D. Similarly, if (5') is satisfied, $(rX_1 + sX_2)(mX_1 + nX_2) = ma(rX_1 + sX_2) = \xi(mX_1 + nX_2)(rX_1 + sX_2)$.

Conversely if ξ is a non-trivial homomorphism of G into D, then multiplication defined by $XY = \xi(X)Y$ for $X, Y \in G$ is associative and distributive with respect to addition. Since ξ is non-trivial, there exists $K \in G$ such that $\xi(K) \neq 0$. Since G has rank 2, there exists $L \in G$ such that K and L are independent. If $KL = LK$, then $\xi(K)L = \xi(L)K$ which implies $\xi(K) = \xi(L) = 0$, which is a contradiction. Hence the multiplication yields a non-commutative ring $\mathfrak{R}$ over G. An analogous discussion can be given for $XY = \xi(Y)X$.

Corollary 1. *If H is a torsion-free abelian group of arbitrary rank and if ξ is a non-trivial homomorphism of H into the nucleus D of H, then multiplication defined by $XY = \xi(X)Y$ or $XY = \xi(Y)X$, for $X, Y \in H$ yields a (non-zero) ring $\mathfrak{R}$ over H. $\mathfrak{R}$ is non-commutative if and only if the rank of H is greater than one.*

Proof. The fact that the given multiplication is well-defined, associative, and distributive with respect to addition does not depend on the rank of H, and hence follows as in Theorem 2.

If $\mathfrak{R}$ is non-commutative, then H cannot have rank 1, since the only (non-zero) rings over torsion-free groups of rank one are isomorphic to subrings of the field of rational numbers ([2], p. 177). Conversely, if the rank of H is at least 2, the fact that $\mathfrak{R}$ is non-commutative follows from the proof of Theorem 2.

Corollary 2. *A non-commutative ring $\mathfrak{R}$ over a torsion-free group of rank* 2 *contains an ideal $\mathfrak{J}$ such that* (i) $\mathfrak{J}^2=0$, *and* (ii) *the additive group $\mathfrak{J}^+$ of $\mathfrak{J}$ has rank* 1. *In particular, $\mathfrak{R}$ contains proper divisors of zero.*

Proof. We note that $\{Y\}$, the pure subgroup of G generated by Y in (5) and (5'), Lemma 2, is an ideal in $\mathfrak{R}$ with the stated properties.

The non-commutative rings $\mathfrak{R}$ over G occur in antiisomorphic pairs, defined by $XY=\xi(X)Y$ and $XY=\xi(Y)X$ in Theorem 2. Thus, to determine the essentially different rings over G, we need only consider those defined by $XY=\xi(X)Y$. We denote such a ring by $(\mathfrak{R}, \xi)$.

Corollary 3. *The ring $(\mathfrak{R}, \xi)$ is isomorphic to the ring $(\mathfrak{R}, \eta)$ if and only if there exists an automorphism φ of G such that $\xi=\eta\varphi$.*

Proof. Let φ be an isomorphism of $(\mathfrak{R},\xi)$ onto $(\mathfrak{R}, \eta)$. Then φ induces an automorphism φ of G. Any automorphism φ of G has the property that if $rX\in G$ for $r\in R$, $X\in G$, then $\varphi\,(rX)=r\varphi(X)$. We have

$$\xi(X)\varphi(Y)=\varphi(XY)=\varphi(X)\varphi(Y)=\eta[\varphi(X)]\varphi(Y)$$

for all $X\in G$. Hence $\xi=\eta\varphi$.

Conversely, if $\xi=\eta\varphi$ for an automorphism φ of G, then it is clear by the above calculation that φ is a ring isomorphism of $(\mathfrak{R}, \xi)$ onto $(\mathfrak{R}, \eta)$.

V. Rings over G without divisors of zero

In this section we consider rings over a torsion-free group G of rank 2 which contain no proper divisors of zero. By Corollary 2 of Theorem 2, such a ring is necessarily commutative. In theorem 3 we characterize those rings $\mathfrak{R}$ over G which contain no proper divisors of zero.

Theorem 3. *Let G be a torsion-free group of rank* 2. *Then $\mathfrak{R}$ is a ring over G without proper divisors of zero if and only if $\mathfrak{R}$ is isomorphic to a subring of a quadratic extension $R(\alpha)$ of R.*

Proof. Let $\mathfrak{R}$ be a ring over G without divisors of zero. Then, as remarked above, $\mathfrak{R}$ is commutative. Hence, by Lemma 1, there exists $X\in G$ such that X and X^2 are independent. Then $X^3=rX+sX^2$, where $r=r_1/r_2$

and $s = s_1/s_2$. Hence $[r_2, s_2]X$ and $([r_2, s_2]X)^2$ are independent elements of G where $([r_2, s_2]X)^3 = [r_2, s_2]^2 r([r_2, s_2]X) + [r_2, s_2]s([r_2, s_2]X)^2$, so that we may assume that X and X^2 are independent elements of G, where $X^3 = aX + bX^2$ with a and b integers.

Consider the polynomial $x^2 - bx - a$, and let α and β be its zeros. Then α and β are not rational. For suppose the contrary. Then $X^4 - bX^3 - aX^2 = 0$, and since $\alpha + \beta = b$ and $\alpha\beta = -a$, we have

$$X^4 - bX^3 - aX^2 = (X^2 - \alpha X)(X^2 - \beta X) = 0.$$

Since α and β are integers, and since X and X^2 are independent, $X^2 - \alpha X$ and $X^2 - \beta X$ are non-zero elements of $\mathfrak{R}$. But this contradicts the hypothesis that $\mathfrak{R}$ has no divisors of zero.

Since α is a zero of $x^2 - bx - a$, we have $\alpha^3 = a\alpha + b\alpha^2$. Hence the correspondence $\varphi: \mathfrak{R} \to R(\alpha)$ defined by $mX + nX^2 \to m\alpha + n\alpha^2$ is a ring homomorphism. Moreover φ is an isomorphism for $m\alpha + n\alpha^2 = 0$ implies $(m + nb)\alpha + na = 0$, and since α is not rational, this yields $m + nb = 0$ and $na = 0$. But again since α is not rational, $a \neq 0$. Hence $n = m = 0$.

Conversely, any subring of a quadratic extension $R(\alpha)$ of R is an integral domain.

In order to derive a necessary condition for the existence of a ring $\mathfrak{R}$ without zero divisors over the group G, we find additional necessary and sufficient conditions that a ring $\mathfrak{R}$ over G has no zero divisors.

Theorem 4. *Let G be a torsion-free group of rank 2. Then $\mathfrak{R}$ is a ring over G without zero divisors if and only if there exists an element $Y \in \mathfrak{R}$ such that*

(i) *Y and Y^2 are independent;*

(ii) *$Y^3 = cY$, where c is a non-square integer.*

Proof. In the proof of Theorem 3, X can be chosen so that $X^3 = aX + bX^2$ where $b = 2q$ is even (since the element X in the proof of Theorem 3 can be replaced by $2X$). Now $\alpha^2 - 2q\alpha - a = 0$ so that we have

$$(\alpha - q)^2 = q^2 + a; \; (a\alpha - aq)^2 = a^2(q^2 + a);$$

$$a\alpha - aq = a\alpha - q(\alpha - q)^2 + q^3 = a\alpha - q(\alpha^2 - 2q\alpha) \tag{6}$$

$$= (a + 2q)\alpha - q\alpha^2;$$

$$[(a + 2q)\alpha - q\alpha^2]^2 = (a\alpha - aq)^2 = a^2(q^2 + a); \tag{7}$$

$$[(a + 2q)\alpha - q\alpha^2]^3 = a^2(q^2 + a)[(a + 2q)\alpha - q\alpha^2]. \tag{8}$$

Now let $Y = (a + 2q)X - qX^2$. Then $\varphi(Y) = (a + 2q)\alpha - q\alpha^2$, where φ is the isomorphism of $\mathfrak{R}$ into $R(\alpha)$ given in Theorem 3. We note first that

Y and Y^2 are independent. For if $eY+fY^2=0$ for integers e and f, we would have $e\varphi(Y)+f\varphi(Y)^2=0$, which by (6) and (7) would yield

$$e(a\alpha-aq)+fa^2(q^2+a)=0.$$

But this implies α is rational unless $e=f=0$.

It is immediate from (8) that $Y^3=a^2(q^2+a)Y$ where $a^2(q^2+a)$ is a non-square integer.

Conversely, suppose that there exists an element $Y\in\mathfrak{R}$ which satisfies (i) and (ii). Assume that $\mathfrak{R}$ has proper divisors of zero, so that there exist non-zero elements $rY+sY^2$ and $uY+vY^2$ such that $(rY+sY^2)(uY+vY^2)=0$. Using (ii), this yields

$$c(rv+su)Y+(ru+svc)Y^2=0.$$

Since Y and Y^2 are independent, we have

$$c(rv+su)=0 \text{ and } ru+svc=0.$$

Since $c\neq 0$ by (ii), and not both r and s are zero, we obtain

$$0=uv-uvc=uv(1-c)$$

which implies (since $c\neq 1$ by (ii)) that $u=0$ or $v=0$. Suppose $u=0$. Then $crv=svc=0$, which implies $r=s=0$, since not both u and v are zero. But this is a contradiction. Similarly, a contradiction is obtained if $v=0$, and this completes the proof of the theorem.

Definition. A torsion-free group H of rank 1 is said to be of *nil type* if the only ring $\mathfrak{R}$ over H is a zero ring.

Let $p_1, p_2, \ldots, p_j, \ldots$ be an enumeration of the primes in their natural order and let k_j be the maximum power of the prime p_j which can occur in the denominator of an element of H. Then we write $H=(k_1, k_2, \ldots, k_j, \ldots)$. It is proved in [2; p. 175] that H is of nil type if and only if there is an infinite number of k_j such that $0<k_j<\infty$.

Theorem 5. *If there exists a ring $\mathfrak{R}$ without zero divisors over the torsion-free group G of rank 2, then G contains independent elements X_1 and X_2 such that the groups of rank 1, $U\supseteq U_0$ and $V\supseteq V_0$, belonging to X_1, X_2 satisfy*

(a) *$U\cong V$ and $U_0\cong V_0$;*

(b) *none of the groups U, U_0, V, V_0 are of nil type.*

Proof. Let X_1 and X_2 be the elements Y and Y^2 satisfying the conditions (i) and (ii) of Theorem 4, and let $U\supseteq U_0$, $V\supseteq V_0$ be the groups of rank 1 belonging to Y, Y^2. Let $Z=rY+sY^2$ be an element of G. Then

$$ZY=rY^2+sY^3=scY+rY^2.$$

Since Z is arbitrary, this implies $cV \subseteq U \subseteq V$, so that U and V are isomorphic [4; p. 210]. It can be shown similarly that $cV_0 \subseteq U_0 \subseteq V_0$ and this completes the proof of (a).

Assume now that U is of nil type. Write $U=(k_1, k_2, \ldots, k_j, \ldots)$ and $V=(h_1, h_2, \ldots, h_j, \ldots)$. Since $U \cong V$, there exist infinitely many $j \neq 1$ (i. e. $p_j \neq 2$) such that $0 < k_j = h_j < \infty$ ([2], p. 171). For any such j, $r = 1/p_j^{k_j} \in U$, and consider

$$\begin{aligned}(rY+sY^2)^2 &= r^2Y^2 + 2rsY^3 + s^2Y^4\\ &= 2rscY + (r^2+cs^2)Y^2.\end{aligned}$$

Let $s=m/n$, $(m, n)=1$, and $n=p_j^l k, (p_j, k)=1$. We have

$$\begin{aligned}r^2+cs^2 &= 1/p_j^{2k_j} + \mathrm{cm}^2/p_j^{2l}k^2\\ &= (p_j^{2l}k^2 + p_j^{2k_j}m^2c)/p_j^{2k_j+2l}k^2 \in V.\end{aligned}$$

Since $k_j = h_j$, we must have that $p_j^{k_j+2l}$ divides $p_j^{2l}k^2 + p_j^{2k_j}m^2c$. If $l < k_j$, then $p_j^{k_j}$ divides $k^2 + p_j^{2k_j-2l}m^2c$, which is a contradiction. Therefore $l \geqq k_j > 0$. Since

$$2rsc = 2mc/p_j^{k_j+l}k \in U,$$

p_j^l divides $2mc$. Hence p_j^l divides c. Since this holds for infinitely many primes p_j, $c=0$, which is a contradiction.

To show that V_0 (and consequently U_0) is not of nil type, we observe that the pure subgroup $\{Y^2\}$ is a subring of $\mathfrak{R}$ and hence is an integral domain. Thus V_0, which is isomorphic to $\{Y^2\}$ is not of nil type.

The authors believe that there is a stronger theorem than Theorem 5, namely that if $\mathfrak{R}$ is a ring without zero divisors over G, then G decomposes as a direct sum $U \oplus V$, where $U \cong V$ and V is not of nil type. This latter condition is sufficient for the existence of a ring $\mathfrak{R}$ without zero divisors over G. To prove this, we suppose that $G = U \oplus U$, where $U=(k_1, k_2, \ldots, k_j, \ldots)$ such that each k_j is either 0 or ∞. Then in the quadratic extension $R(\sqrt{n})$, the set $[a+b\sqrt{n} \mid a, b \in U]$ is a subring of $R(\sqrt{n})$ with additive group $G = U \oplus U$.

In conclusion it should be remarked that a ring over a torsion-free group G can be imbedded as a subring in an algebra over the field R of rational numbers of dimension equal to the rank of G. This is accomplished by imbedding G in the tensor product $G \otimes R$, where G and R are regarded as modules over the integers. Then $G \otimes R$ is a vector space over the field R in a natural way, and for $g \otimes r \in G \otimes R$, $r \neq 0$, $r^{-1}(g \otimes r) = g \otimes 1 \in G$. For G of rank 2, if one computes all algebras of dimension 2 over R, then the possible multiplication tables for independent elements of G are obtained from the multiplication tables for the algebras. By observing which algebras are non-commutative and which do not have zero divisors, one obtains Lemma 2, Theorem 3, and Theorem 4 by this alternate method.

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(Received October 27, 1958)